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SOME PROPERTIES OF THE NATURAL FREQUENCIES OF ELECTROELASTIC BODIES OF BOUNDED DIMENSIONS†

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Homogeneous problems of the natural oscillations of bounded electroelastic solids, in contact with rigid plane punches and coated with a system of open- and short-circuited electrodes, are considered. A variational principle is constructed which has the properties of minimality, similar to the well-known variational principle [1] for problems with only short-circuited electrodes. The discreteness of the spectrum and the completeness of the eigenfunctions are proved. As a consequence of variational principles, the properties of an increase or a decrease in the natural frequencies when the mechanical and electrical boundary conditions and the moduli of the electroelastic solid change are established. It is noted that changes in the mechanical and electrical parameters cause opposite changes in the natural frequencies. A confirmation of the fact that, for an electroelastic solid with a multi-electrode coating, the natural frequencies for short-circuited electrodes (the electric resonance frequencies) do not exceed the corresponding frequencies for open-circuited electrodes (the antiresonance frequencies) is obtained as a special case of one of the theorems proved.

1. FORMULATION OF THE PROBLEM

Suppose an electroelastic solid occupies a region Ω, bounded in R^3. We will assume that the region Ω and its boundary $\partial\Omega = S$ are subject to the following conditions: Ω is the sum of a finite number of sets, star-shaped with respect to any spheres contained in them, while S is a Lipschitz boundary of class C^1. These (Ω, S) conditions are presented more comprehensively in [1].

Confining ourselves to investigating steady $\exp(i\omega t)$ oscillating modes, we will use only the amplitude values for all the physical quantities considered without additional provisos.

The homogeneous problems of the oscillations of electroelastic solids include differential equations in Ω consisting of the field equations in the electrostatics approximations

$$-\nabla \cdot \boldsymbol{\sigma} = \rho\omega^2 \mathbf{u} \quad (1.1)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (1.2)$$

and the governing relations of a linear electroelastic solid

$$\boldsymbol{\sigma} = \mathbf{c}^E \cdot\cdot\, \boldsymbol{\epsilon} - \mathbf{e}^T \cdot \mathbf{E} \quad (1.3)$$

$$\mathbf{D} = \mathbf{e} \cdot\cdot\, \boldsymbol{\epsilon} + \mathbf{Э}^S \cdot \mathbf{E} \quad (1.4)$$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2, \quad \mathbf{E} = \mathbf{E}(\varphi) = -\nabla\varphi \quad (1.5)$$

We have used the notation that is standard [2] for the theory of electroelasticity, namely, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are second-rank stress and strain tensors, $\mathbf{D}$ and $\mathbf{E}$ are the electric induction and electric-field vectors, ρ is the density of the material, ω is the angular frequency of the oscillations, $\mathbf{u}$ is the displacement vector, φ is the electric-field potential, $\mathbf{c}^E$ is a fourth-rank tensor of the elastic moduli, $\mathbf{e}$ is a third-rank tensor of the piezoelectric moduli, $Э^s$ is a second-rank permitivity tensor and $(\ldots)^T$ is the operation of transposition.

We will assume that the function $\rho(x)$ is a piecewise-continuous and $\rho(\mathbf{x}) \geqslant \rho_0 > 0$, the components of the tensors $\mathbf{c}^E$, $\mathbf{e}$, $Э^S$ are piecewise-continuous together with their first derivatives with respect to x, where $c^E_{ijkl} = c^E_{ijlk} = c^E_{jikl} = c^E_{klij}$, $e_{ijk} = c^E_{ikj}$, $, Э^s_{ij} = Э^s_{ji}$ and c^E_{ijkl} and $Э^s_{ij}$ satisfy the conditions for them to be strictly positive-definite.

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The boundary conditions are of two types: mechanical and electrical.

To formulate the mechanical boundary conditions we will assume that the boundary S can be split into two subsets: S_σ and S_u ($S = S_\sigma \cup S_u$).

The parts of the boundary S_σ are stress-free, i.e.

$$\mathbf{n} \cdot \boldsymbol{\sigma} = 0, \quad \mathbf{x} \in S_\sigma \tag{1.6}$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface.

Suppose $S_u = \cup S_{uk}$, $k = 0, 1, \ldots, L$; $S_{u0} \neq \varnothing$, S_{uk} do not border one another, while among S_{uk} there are $L + 1 - l$ rigidly clamped sections and l plane parts, in contact with plane punches, whose masses will be neglected. We will assume, for simplicity, that all these l plane parts are perpendicular to the x_3 axis. Then, we can assume the following boundary conditions on S_{uk}

$$\boldsymbol{\sigma} \cdot \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n})\mathbf{n} = 0, \quad \mathbf{x} \in S_{ui}, \quad i = 1, 2, \ldots, l, \quad l \leqslant L \tag{1.7}$$

$$\mathbf{u} \cdot \mathbf{n} = \sum_{j=0}^{2} \alpha_{ji}^{u} x_j, \quad \mathbf{x} \in S_{ui}, \quad x_0 = 1, \quad i = 1, 2, \ldots, l, \quad l \leqslant L \tag{1.8}$$

$$\int_{S_{ui}} x_j \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = 0, \quad x_0 = 1, \quad j = 0, 1, 2, \quad i = 1, 2, \ldots, l, \quad l \leqslant L \tag{1.9}$$

$$\mathbf{u} = 0, \quad \mathbf{x} \in S_{uj}, \quad j = 0, l+1, l+2, \ldots, L, \quad S_{u0} \neq \varnothing \tag{1.10}$$

where the contact conditions (1.7)–(1.9) do not occur when $l = 0$.

The unknown quantities α_{ji}^{u}, which specify the plane displacements of the sections S_{ui}, are to be determined from the integral conditions (1.9).

To specify the electrical boundary conditions we will assume that the boundary S is split into two subsets: S_D and S_φ. The sections S_D do not have electrodes, and the following conditions are satisfied on them

$$\mathbf{n} \cdot \mathbf{D} = 0, \quad \mathbf{x} \in S_D \tag{1.11}$$

The subset S_φ is a combination of $M + 1$ sections $S_{\varphi k}$ ($k = 0, 1, \ldots, M$), which are not adjacent to one another, coated with infinitesimally thin electrodes. We will specify the following boundary conditions on these sections

$$\varphi = \Phi_i, \quad \mathbf{x} \in S_{\varphi i}, \quad i = 1, 2, \ldots, m, \quad m \leqslant M, \quad \Phi_i = \text{const} \tag{1.12}$$

$$\int_{S_{\varphi i}} \mathbf{n} \cdot \mathbf{D} dS = 0, \quad \mathbf{x} \in S_{\varphi i}, \quad i = 1, 2, \ldots, m, \quad m \leqslant M \tag{1.13}$$

$$\varphi = 0, \quad \mathbf{x} \in S_{\varphi j}, \quad j = 0, m+1, m+2, \ldots, M, \quad S_{\varphi 0} \neq \varnothing \tag{1.14}$$

By (1.12) and (1.13) there are m open-circuited electrodes on which the potentials Φ_i are initially unknown, but the overall charges on each electrode are equal to zero. The remaining $M + 1 - m$ electrodes are assumed to be short-circuited with zero values of the potentials. The cases $m = 0$ and $m = M$ are not ignored. In the first case, conditions (1.12) and (1.13) are not present, and all the electrodes are short-circuited. In the second case, condition (1.14) only applies for the electrode $S_{\varphi 0}$, and all the electrodes $S_{\varphi k}$ ($k = 0, 1, \ldots, m = M$) can be assumed to be open-circuited.

In fact, since the potential φ is defined apart from a constant, we can take (1.14) for $S_{\varphi 0}$, and from the equation

$$\int_S \mathbf{n} \cdot \mathbf{D} dS = 0$$

which follows from (1.2), and from (1.11) and (1.13) it follows that (1.13) is also satisfied for $S_{\varphi 0}$ with $\Phi_0 = 0$.

We will also assume that all the sections S_{uk} and $S_{\varphi k}$ have Lipschitz boundaries of the class C^1 [1] and do not intersect one another.

Problem (1.1)–(1.14) is a problem of natural oscillations and consists of finding the eigenvalues ω^2 and eigenfunctions $\mathbf{u}$ and φ, which give non-trivial solutions of the homogeneous boundary-value problem.

Conditions (1.12) and (1.13) are similar to the mechanical contact conditions (1.7)–(1.9) with rigid punches. These "contact" boundary conditions distinguish this problem from the problem investigated previously in [1] with only short-circuited electrodes with $m = 0$ and $l = 0$.

We will establish the mathematical properties of the spectrum of problem (1.1)–(1.14) by combining the approaches used in [1, 3] when investigating dynamic problems of electroelasticity and contact problems of the theory of elasticity.

2. GENERALIZED AND VARIATIONAL FORMULATION OF THE PROBLEM

We will introduce the space of functions φ and the vector functions $\mathbf{u}$, defined on Ω, which we shall need later.

We will denote by H_ρ the space of vector functions $\mathbf{u} \in L_2$ with the scalar product

$$(\mathbf{u}, \mathbf{v})_\rho = \int_\Omega \rho \mathbf{u} \cdot \mathbf{v} d\Omega$$

On the set of vector functions $\mathbf{u} \in C^1$ which satisfy (1.10) and (1.8) for arbitrary α_{ji}^u on S_{ui}, we will introduce the scalar products

$$(\mathbf{u}, \mathbf{v})_{ul} = \int_\Omega (\nabla \mathbf{u}) \cdot\cdot (\nabla \mathbf{v})^T d\Omega \tag{2.1}$$

The closure of this set of vector functions $\mathbf{u}$ in the norm generated by the scalar product (2.1) will be denoted by H_{ul}.

On the set of functions $\varphi \in C^1$ which satisfy (1.14) and (1.12) for arbitrary Φ_i on $S_{\varphi i}$ ($i = 1, 2, \ldots, m$), we will introduce the scalar products

$$(\varphi, \psi)_{\varphi m} = \int_\Omega \nabla\varphi \cdot \nabla\psi d\Omega \tag{2.2}$$

The closure of this set in the norm generated by the scalar product (2.2) will be denoted by $H_{\varphi m}$.

Then, for arbitrary $\mathbf{v} \in H_{ul}$, $\chi \in H_{\varphi m}$, using standard procedures, we can convert (1.1)–(1.14) to the form

$$-\omega^2 \rho(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{v}) + e(\varphi, \mathbf{v}) = 0 \tag{2.3}$$

$$-e(\chi, \mathbf{u}) + \ni(\varphi, \chi) = 0 \tag{2.4}$$

where

$$\begin{gathered} \rho(\mathbf{u}, \mathbf{v}) = \int_\Omega \rho \mathbf{u} \cdot \mathbf{v} d\Omega, \quad c(\mathbf{u}, \mathbf{v}) = \int_\Omega \boldsymbol{\epsilon}(\mathbf{u}) \cdot\cdot \mathbf{c}^E \cdot\cdot \boldsymbol{\epsilon}(\mathbf{v}) d\Omega \\ e(\varphi, \mathbf{v}) = \int_\Omega \nabla\varphi \cdot \mathbf{e} \cdot\cdot \boldsymbol{\epsilon}(\mathbf{v}) d\Omega, \quad \ni(\varphi, \chi) = \int_\Omega \nabla\varphi \cdot \boldsymbol{\ni}^S \cdot \nabla\chi d\Omega \end{gathered} \tag{2.5}$$

By virtue of the properties assumed earlier, the forms $\rho(\mathbf{u}, \mathbf{v})$, $c(\mathbf{u}, \mathbf{v})$ and $\ni(\varphi, \chi)$ in (2.5) are symmetrical, bilinear and positive-definite in L_2, H_{ul} and $H_{\varphi m}$, respectively, while $e(\varphi, \mathbf{v})$ is a bilinear form.

Since for fixed $\mathbf{u} \in H_{ul}$, $\varphi \in H_{\varphi m}$, $e(\chi, \mathbf{u})$ and $\ni(\varphi, \chi)$ are linear-bounded functions in $H_{\varphi m}$, by Riesz' theorem elements $e\mathbf{u}$, $\ni\varphi \in H_{\varphi m}$ exist and are unique such that for $\chi \in H_{\varphi m}$

$$e(\chi, \mathbf{u}) = (\chi, e\mathbf{u})_{\varphi m} \tag{2.6}$$

$$\ni(\varphi, \chi) = (\ni\varphi, \chi)_{\varphi m} \tag{2.7}$$

It is obvious that $e\mathbf{u}$ and $\ni\varphi$ are linear bounded operators acting from H_{ul} into $H_{\varphi m}$ and from $H_{\varphi m}$ into $H_{\varphi m}$, respectively, and an inverse exists for the operator $\ni\varphi$.

From (2.4), (2.6) and (2.7) we obtain that

$$\ni\varphi = e\mathbf{u},\ \ \varphi = A_{lm}\mathbf{u},\ \ A_{lm} = \ni^{-1}e \tag{2.8}$$

where the operator A_{lm} acts from H_{ul} into $H_{\varphi m}$, and is linear and bounded.

Using (2.6)–(2.8) we can represent system (2.3), (2.4) in the form

$$-\omega^2\rho(\mathbf{u}, \mathbf{v}) + c\ni_{lm}(\mathbf{u}, \mathbf{v}) = 0 \tag{2.9}$$

where

$$c\ni_{lm}(\mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v}) + \ni(A_{lm}\mathbf{u}, A_{lm}\mathbf{v}) \tag{2.10}$$

Definition. We will call the triple of quantities (ω^2, $\mathbf{u} \in H_{ul}$, $\varphi \in H_{\varphi m}$), which satisfy (2.9) and (2.8) for arbitrary vector functions $\mathbf{v} \in H_{ul}$ or, which is equivalent (2.3) and (2.4) for arbitrary $\mathbf{v} \in H_{ul}$, $\chi \in H_{\varphi m}$, a generalized solution of problem (1.1)–(1.14).

By discussions similar to those presented in [1] for the case when $m = 0$ and $l = 0$, we can show that the space $H_{c\ni lm}$, which is the closure of the set of vector functions $\mathbf{u} \in C^1$, satisfying (1.8) and (1.10) in the norm generated by the scalar product (2.10), is equivalent to H_{ul}, and the following theorem follows from the complete continuity of the operator of embedding from H_{ul} into H_ρ, as also in the general situation [4].

Theorem 2.1. The operator equation (2.9) has a discrete spectrum $0 < \omega_{lm1}^2 \leqslant \omega_{lm2}^2 \leqslant \ldots \leqslant \omega_{lmk}^2 \leqslant \ldots,\ \omega_{lmk}^2 \to \infty$ as $k \to \infty$, and the corresponding eigenfunctions $\mathbf{u}_{lm}^{(k)}$ form a system that is orthogonal and complete in the spaces H_ρ and $H_{c\ni lm}$.

Theorem 2.2. (The Courant–Fisher minimax principle)

$$\omega_{lmk}^2 = \max_{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k-1} \in H_{ul}} \left[\min_{\substack{\mathbf{v} \neq 0, \mathbf{v} \in H_{ul} \\ \rho(\mathbf{v}, \mathbf{w}_j) = 0,\ j = 1, 2, \ldots, k-1}} R_{lm}(\mathbf{v}) \right]$$

where $R_{lm}(v)$ is the Rayleigh quotient

$$R_{lm}(\mathbf{v}) = \frac{c\ni_{lm}(\mathbf{v}, \mathbf{v})}{\rho(\mathbf{v}, \mathbf{v})}$$

The subscripts l and m in Theorems 2.1 and 2.2 are introduced for convenience in indicating later that the generalized solutions belong to problems with l sections in contact with rigid plane punches, and with m open-circuited electrodes.

Note that Theorem 2.1 is important for justifying Fourier's method of solving unsteady problems of electroelasticity [5, 6].

3. THE RELATION BETWEEN THE OPERATORS A_{lm} AND A_{l0}

We will introduce the functions $\psi_j \in H_{\varphi M}$ ($j = 1, 2, \ldots, M$), $\psi_j = \delta_{jk}$, $\mathbf{x} \in S_{\varphi k}$ ($k = 0, 1, \ldots, M$) which, for all $\chi \in H_{\varphi 0}$, satisfy the integral identity

$$\ni(\psi_j, \chi) = 0 \tag{3.1}$$

Using the quantities $C_{ij}^M = \ni(\psi_i, \psi_j)$ we form the $M \times M$ matrix $\mathbf{C}^M$ called the static-capacitance matrix in electrostatics. In view of the fact that the form $\ni(\varphi, \chi)$ is symmetric and positive-definite in the space $H_{\varphi m}$, the matrix $\mathbf{C}^M$ will also be symmetric and positive-definite. Then, any principal $m \times m$ submatrices $\mathbf{C}^m$, made up of the elements C_{ij}^m, $1 \leqslant i, j \leqslant m$ will also be symmetric and positive-definite, as also the inverse matrices $\mathbf{S}^m = (\mathbf{C}^m)^{-1}$.

For a problem with m open-circuited electrodes, the arbitrary element $\chi \in H_{\varphi m}$ and the solution

$\varphi \in \mathrm{H}_{\varphi m}$ can be represented in the form

$$\chi = \chi_0 + \sum_{k=1}^{m} X_k \psi_k, \quad \varphi = \varphi_0 + \sum_{k=1}^{m} \Phi_k \psi_k \tag{3.2}$$

where $\chi_0, \varphi_0 \in \mathrm{H}_{\varphi 0}$, X_k are arbitrary constants, while Φ_k are initially unknown values of φ on $S_{\varphi k}$ from (1.12).

Substituting (3.2) into (2.4) and using (3.1) and the fact that χ_0, X_k, $k = 1, 2, \ldots, m$ are arbitrary, we obtain

$$\partial(\varphi_0, \chi_0) = e(\chi_0, \mathbf{u}) \tag{3.3}$$

$$e(\psi_k, \mathbf{u}) = \sum_{j=1}^{m} C_{kj}^m \Phi_j \tag{3.4}$$

From (3.3) by (2.6)–(2.8) we have $\varphi_0 = A_{l0}\mathbf{u}$, and from (3.4) we have

$$\Phi_k = \sum_{j=1}^{m} S_{kj}^m \Psi_j^u, \quad \Psi_j^u = e(\psi_j, \mathbf{u}) \tag{3.5}$$

As a result, using (2.8) and (3.5) from (3.2) for φ we obtain the following relation between the operators A_{lm} and A_{l0}

$$A_{lm}\mathbf{u} = A_{l0}\mathbf{u} + \sum_{k=1}^{m}\sum_{j=1}^{m} S_{kj}^m \Psi_j^u \psi_k$$

Since $A_{l0}\mathbf{u} \in \mathrm{H}_{\varphi 0}$, using property (3.1) we will also have

$$\partial(A_{lm}\mathbf{u}, A_{lm}\mathbf{v}) = \partial(A_{l0}, \mathbf{u}, A_{l0}\mathbf{v}) + \sum_{k=1}^{m}\sum_{j=1}^{m} S_{kj}^m \Psi_k^u \Psi_j^v \tag{3.6}$$

for any $\mathbf{u}, \mathbf{v} \in \mathrm{H}_{ul}$.

4. CONSEQUENCE OF THE VARIATIONAL FORMATIONS

We will call problem (1.1)–(1.4) the lm-problem, emphasizing by this the presence of l parts of S_{ui}, in contact with plane punches, and m open-circuited electrodes $S_{\varphi i}$.

We will consider two similar lm- and pm-problems, which differ solely in the numbers l and p of contacting parts of S_{ui} in (1.7)–(1.10). All the remaining governing parameters from (1.1)–(1.14) in the lm- and lp-problems are assumed to be the same.

Theorem 4.1. If $0 \leqslant l < p \leqslant L$, for any k the kth natural frequency of the lm-problem is no less than the kth natural frequency of the pm-problem, i.e. $\omega_{lmk}^2 \geqslant \omega_{pmk}^2$.

Since $l < p$, we have $\mathrm{H}_{ul} \subset \mathrm{H}_{up}$, and for all $\mathbf{v} \in \mathrm{H}_{ul} A_{lm}\, \mathbf{v} = A_{pm}\mathbf{v}$ and $R_{lm}(\mathbf{v}) = R_{pm}(\mathbf{v})$.

Then, Theorem 4.1 follows from the well-known discussions in [4], using Theorem 2.2.

We will now consider two similar lm- and ln-problems (1.1)–(1.14), which differ solely in the numbers m and n of open-circuited electrodes $S_{\varphi i}$ in (1.12)–(1.14).

Theorem 4.2. If $0 \leqslant m < n \leqslant M$, then, for any k, the kth natural frequency of the lm-problem does not exceed the kth natural frequency of the ln-problem, i.e. $\omega_{lmk}^2 \leqslant \omega_{lnk}^2$.

By (2.10) and (3.6) for $m \neq 0$ we have

$$c\partial_{ln}(\mathbf{v},\mathbf{v}) - c\partial_{lm}(\mathbf{v},\mathbf{v}) = \sum_{k=1}^{n}\sum_{j=1}^{n} S_{kj}^n \Psi_k^v \Psi_j^v - \sum_{k=1}^{m}\sum_{j=1}^{m} S_{kj}^m \Psi_k^v \Psi_j^v \tag{4.1}$$

We will represent the matrix $\mathbf{S}^n$ in partitioned form

$$\mathbf{S}^n = \begin{vmatrix} \mathbf{G} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{F} \end{vmatrix} \tag{4.2}$$

where $\mathbf{G}$ and $\mathbf{F}$ are $m \times m$ and $(n-m) \times (n-m)$ positive-definite symmetric matrices, respectively. Since the matrix $\mathbf{S}^n$ is the inverse of $\mathbf{C}^n$, while $\mathbf{S}^m$ is the inverse of the submatrix $\mathbf{C}^m$ of the matrix $\mathbf{C}^n$, we have from the representation [7] for a matrix that is the inverse of a partitioned matrix

$$\mathbf{S}^m = \mathbf{G} - \mathbf{B}^T \cdot \mathbf{F}^{-1} \cdot \mathbf{B} \tag{4.3}$$

Using (4.2) and (4.3) the right-hand side of (4.1) can be represented in the form

$$(\mathbf{\Psi}^{\upsilon})^T \cdot \begin{vmatrix} \mathbf{B}^T \cdot \mathbf{F}^{-1} \cdot \mathbf{B} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{F} \end{vmatrix} \cdot \mathbf{\Psi}^{\upsilon} \tag{4.4}$$

where $\mathbf{\Psi}^{\upsilon}$ is an n-dimensional vector with components Ψ_k^{υ}.

The matrix in (4.4) is positive-semidefinite [7], and consequently for all $\mathbf{v} \in \mathrm{H}_{ul}$ the following inequalities hold

$$c\vartheta_{lm}(\mathbf{v}, \mathbf{v}) \leqslant c\vartheta_{ln}(\mathbf{v}, \mathbf{v}), \quad R_{lm}(\mathbf{v}) \leqslant R_{ln}(\mathbf{v})$$

These inequalities also hold when $m = 0$, since, in this case, there are no sums with S_{kj}^m in (4.1), and the matrix S^n is positive-definite.

The inequality proved for the Rayleigh quotients, taking Theorem 2.2 into account, in fact also proves Theorem 4.2 [4].

We will investigate the change in the natural frequencies of the lm-problem (1.1)–(1.14) when some of its parameters change. These changes will be indicated explicitly in the formulations of the following theorems, and all the quantities referring to the modified m-problems will be indicated by an asterisk. As above, for the initial and modified problems all the theorems not indicated in the formulations which define the parameters from (1.1)–(1.14) are assumed to be identical.

Theorem 4.3. If $S_u \supset S_{*u}$, $S_{uj} \supset S_{*uj}$, $j = 0, 1, \ldots, L$, we have $\omega_{lmk}^2 \geqslant \omega_{*lmk}^2$ for all k.

Theorem 4.4. If the elastic moduli and the densities of the two lm-problems are such that $c(\mathbf{v}, \mathbf{v}) \geqslant c_*(\mathbf{v}, \mathbf{v})$, $\rho(\mathbf{v}, \mathbf{v}) \leqslant \rho_*(\mathbf{v}, \mathbf{v})$ for any $\mathbf{v} \in \mathrm{H}_{ul}$, then $\omega_{lmk}^2 \geqslant \omega_{*lmk}^2$ for all k.

If $S_u \supset S_{*u}$, $S_{uj} \supset S_{*uj}$, then $\mathrm{H}_u \subset \mathrm{H}_{*u}$ and $A_{lm}(\mathbf{v}) = A_{*lm}(\mathbf{v})$ for all $\mathbf{v} \in \mathrm{H}_{ul}$. Consequently, $R_{lm}(\mathbf{v}) \geqslant R_{*lm}(\mathbf{v})$ for all $\mathbf{v} \in \mathrm{H}_{ul}$ in the conditions of both theorems. This inequality proves Theorems 4.3 and 4.4.

Note. The influence of S_u, the elastic moduli and the density of the electroelastic material on the first natural frequency ω_0^2 of problem (1.1)–(1.11) for $l = 0$ and $m = 0$ was considered in [8]. To do this the following variational principle was employed: $\omega_{001}^2 = \min R(\mathbf{v})$, $\mathbf{v} = 0$ on S_u, and $R(\mathbf{v}) = (c(\mathbf{v}, \mathbf{v}) + \ni(\chi, \chi))/\rho(\mathbf{v}, \mathbf{v})$ when χ is defined in terms of $\mathbf{v}$ as the solution of problem (1.2), (1.4), (1.5), (1.11), (1.14) with φ replaced by χ and $\mathbf{u}$ replaced by $\mathbf{v}$. Note that this formulation is not mathematically rigorous since χ is defined as the classical solution, while for ω^2 and $\mathbf{u}$ a variational approach is used which gives a generalized solution. The condition $c_{ijkl}^E \geqslant c_{*ijkl}^E$ proposed in [8] instead of $c(\mathbf{v}, \mathbf{v}) \geqslant c_*(\mathbf{v}, \mathbf{v})$ in Theorem 4.4 is neither necessary nor sufficient. The correct condition of Theorem 4.4 requires that for the differences of the moduli $\Delta c_{ijkl} = c_{ijkl}^E - c_{*ijkl}^E$, the form $\Delta c(\mathbf{v}, \mathbf{v})$ should be positive-definite, which is not ensured by the inequalities $c_{ijkl}^E \geqslant c_{*ijkl}^E$, and these inequalities are not necessarily satisfied when the form $\Delta c(\mathbf{v}, \mathbf{v})$ is positive-definite.

Theorem 4.5. If $S_\varphi \supset S_{*\varphi}$, $S_{\varphi j} \supset S_{*\varphi j}$, $j = 0, 1, \ldots, M$ we have $\omega_{lmk}^2 \geqslant \omega_{*lmk}^2$ for all k.

If $S_\varphi \supset S_{*\varphi}$, $S_{\varphi j} \supset S_{*\varphi j}$, we have $\mathrm{H}_{\varphi m} \subset \mathrm{H}_{*\varphi m}$. From (2.4) and (2.8) for arbitrary functions $\mathbf{u} \in \mathrm{H}_{ul}$ we have

$$e(\chi, \mathbf{u}) = \vartheta(\varphi, \chi), \quad \varphi, \chi \in \mathrm{H}_{\varphi m}, \quad \varphi = A_{lm}\mathbf{u}$$

$$e(\chi, \mathbf{u}) = \vartheta(\varphi_*, \chi), \quad \varphi_*, \chi \in \mathrm{H}_{*\varphi m}, \quad \varphi_* = A_{*lm}\mathbf{u}$$

Substituting $\chi = \varphi$ into these equations we obtain $\ni(\varphi_*, \varphi)$. Consequently, $0 \leqslant \ni(\varphi - \varphi_*, \varphi - \varphi_*) = \ni(\varphi_*, \varphi_*) - \ni(\varphi, \varphi)$, which implies the inequality $R_{lm}(\mathbf{v}) \leqslant R_{*lm}$ for all $\mathbf{v} \in \mathrm{H}_{ul}$, which proves Theorem 4.5.

Theorem 4.6. If the permittivities of the two lm-problems are such that $\ni(\psi, \psi) \geqslant \ni_*(\psi, \chi)$ for any $\psi \in \mathrm{H}_{\varphi m}$, we have $\omega_{lmk}^2 \leqslant \omega_{*lmk}^2$ for all k.

Since $\ni_{(*)}(\psi, \chi)_{\varphi m} = \ni_{(*)}(\psi, \chi)$ for all $\chi \in \mathrm{H}_{\varphi m}$, we have $\nabla \ni_{(*)}\psi = \ni_{(*)}^s \cdot \nabla\psi$. Then, for any φ,

$\psi \in H_{\varphi m}$ the following chain of inequalities holds

$$(\ni_* \ni \varphi, \psi)_{\varphi m} = (\ni \varphi, \ni_* \psi)_{\varphi m} = \int\limits_\Omega \nabla \ni \varphi \cdot \nabla \ni_* \psi d\Omega =$$

$$= \int\limits_\Omega \nabla \varphi \cdot \ni^S \cdot \ni_*^S \cdot \nabla \psi d\Omega = \int\limits_\Omega \nabla \varphi \cdot \ni_*^S \cdot \ni^S \cdot \nabla \psi d\Omega = (\ni \ni_* \varphi, \psi)_{\varphi m}$$

by virtue of the symmetry of the tensor $\ni^S \cdot \ni^S{}_*$. From the established commutivity of the symmetric positive-definite operators $\ni_*$ and $\ni$ it follows that their quadratic roots also commute [9].

Further, for all $\varphi \in H_{\varphi m}$ we have

$$\ni(\varphi, \varphi) = (\ni \varphi, \varphi)_{\varphi m} = (\ni^{1/2} \varphi, \ni^{1/2} \varphi)_{\varphi m} = (\ni_*^{-1/2} \ni_*^{1/2} \ni^{1/2} \varphi, \ni_*^{-1/2} \ni_*^{1/2} \ni^{1/2} \varphi)_{\varphi m} =$$

$$= (\ni_*^{-1} (\ni_*^{1/2} \ni^{1/2} \varphi), \quad \ni_*^{1/2} \ni^{1/2} \varphi)_{\varphi m}$$

Similarly

$$\ni_*(\varphi, \varphi) = (\ni^{-1} (\ni^{1/2} \ni_*^{1/2} \varphi), \quad \ni^{1/2} \ni_*^{1/2} \varphi)_{\varphi m}$$

Since $\ni_*^{1/2} \ni^{1/2} = \ni^{1/2} \ni_*^{1/2}$, it follows from the condition of the theorem that

$$(\ni^{-1} \psi, \psi)_{\varphi m} \leqslant (\ni_*^{-1} \psi, \psi)_{\varphi m} \tag{4.5}$$

for all $\psi \in H_{\varphi m}$.

Finally, from (2.7) and (2.8) we have

$$\ni_{(*)}(A_{(*)lm} \mathbf{v}, \quad A_{(*)lm} \mathbf{v}) = (\ni_{(*)}^{-1} e \mathbf{v}, e \mathbf{v})_{\varphi m} \tag{4.6}$$

and hence, from (4.5) and (4.6) we have

$$\ni(A_{lm} \mathbf{v}, \quad A_{lm} \mathbf{v}) \leqslant \ni_*(A_{*lm} \mathbf{v}, \quad A_{*lm} \mathbf{v})$$

for all $\mathbf{v} \in H_{ul}$, which in fact also proves the theorem.

Theorem 4.7. If the piezoelectric moduli of the two lm-problems are such that $e_*(\chi, \mathbf{v}) = \lambda e(\chi, \mathbf{v})$ for any $\chi \in H_{\varphi m}$, $\mathbf{v} \in H_{ul}$ and $\lambda \geqslant 1$, we have $\omega_{lmk}^2 \leqslant \omega_{*lmk}^2$ for all k.

Since $\lambda e(\chi, \mathbf{v}) = e(\chi, \lambda \mathbf{v})$, we have $e_* \mathbf{v} = e \lambda \mathbf{v} = \lambda e \mathbf{v}$, and from (4.6) we have $\ni(A_{*lm} \mathbf{v}, A_{*lm} \mathbf{v}) = \lambda^2 (\ni^{-1} e \mathbf{v}, e \mathbf{v})_{\varphi m} = \lambda^2 \ni$, which also proves the theorem.

Note that the conditions of Theorems 4.4, 4.6 and 4.7 for piezoelectric ceramics polarized in the directions of the x_3 axis are satisfied in $\rho \leqslant \rho_*$, $\Delta c_{jj}^E \geqslant 0, j = 1, 3, 4$, $\Delta c_{11}^E \geqslant |\Delta c_{12}^E|$, $\Delta c_{33}^E(\Delta c_{11}^E + \Delta c_{12}^E) \geqslant 2(\Delta c_{13}^E)^2$, $\Delta c_{kl}^E = c_{kl}^E - c_{*kl}^E$ for Theorem 4.4, $\ni_{jj}^s \geqslant \ni_{*jj}^s$ $(j = 1, 3)$ for Theorem 4.6 and $e_{*13} = \lambda e_{13}$, $e_{*15} = \lambda e_{15}$, $e_{*33} = \lambda e_{33}$, $\lambda \geqslant 1$ for Theorem 4.7. (Here we have used the Voigt double-subscript notation for c_{ijkl}^E and e_{ijk}.)

5. FUNDAMENTAL CONCLUSIONS

We will summarize the results of Theorems 4.1–4.6. If on certain parts of S_{uk} we replace the conditions of clamping (1.10) by the contact conditions (1.7)–(1.9), then, by Theorem 4.1, the natural frequencies are reduced.

If on certain parts of $S_{\varphi k}$ we replace the conditions for the potential (1.14) to be zero by electric conditions of the contact type (1.12) and (1.13), then by Theorem 4.2 the natural frequencies are increased.

Note that the natural frequencies of the problem in all the short-circuited electrodes are usually electric resonance frequencies, while the natural frequencies of the problem with all open-circuited electrodes are antiresonance frequencies. Theorem 4.2 therefore also asserts that the antiresonance frequencies are not less than the resonance frequencies with the same order numbers.

By Theorems 4.3 and 4.4 a reduction in the boundary S_u or a reduction in the elastic moduli and an increase in the density lead to a reduction in the natural frequencies. Conversely, by Theorems 4.5

and 4.6 a reduction in the electrode boundary S_φ or a reduction in the permittivities lead to an increase in the natural frequencies.

Comparing the effects reflected in Theorems 4.1, 4.3, 4.4 and 4.2, 4.5 and 4.6 we can conclude that similar changes in the mechanical and electric boundary conditions or in the elastic moduli and the permittivities lead to opposite changes in the natural frequencies.

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